WOLFF'S PROBLEM OF IDEALS IN THE MULTIPLER ALGEBRA ON WEIGHTED DIRICHLET SPACE

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Abstract. We establish an analogue of Wolff's theorem on ideals in $H^\infty(\mathbb{D})$ for the multiplier algebra of weighted Dirichlet space.

In this paper we wish to extend a theorem of Wolff, concerning ideals in $H^\infty(\mathbb{D})$, to the setting of multiplier algebras on weighted Dirichlet spaces. Our techniques will closely follow those used in Banjade-Trent [BT] for the (unweighted) Dirichlet space. The new material requires the boundedness of a certain singular integral operator (Lemma 3) and the boundedness of the Beurling transform (Lemma 4) on some L^2 spaces with weights.

In 1962 Carleson [C] proved his famous "Corona theorem" characterizing when a finitely generated ideal in $H^\infty(\mathbb{D})$ is actually all of $H^\infty(\mathbb{D})$. Independently, Rosenblum [R], Tolokonnikov [To], and Uchiyama gave an infinite version of Carleson's work on $H^\infty(\mathbb{D})$. In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(\mathbb{D})$, Wolff [G] proved the following version:

Theorem A (Wolff). *If*

$$\{f_j\}_{j=1}^n \subset H^\infty(\mathbb{D}), H \in H^\infty(\mathbb{D}) \quad \textit{and}$$

$$|H(z)| \le \left(\sum_{j=1}^n |f_j(z)|^2\right)^{\frac{1}{2}} \quad \textit{for all } z \in \mathbb{D}, \tag{1}$$

then

$$H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n),$$

the ideal generated by $\{f_j\}_{j=1}^n$ *in* $H^\infty(\mathbb{D})$.

It is known that (1) is not, in general, sufficient for H itself or even for H^2 to be in $\mathcal{I}(\{f_j\}_{j=1}^n)$, see Rao [G] and Treil [T].

For the algebra of multipliers on Dirichlet space, the analogue of Wolff's ideal theorem was established by the authors in [BT]. Since

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the analogue of the corona theorem for the algebra of multipliers on weighted Dirichlet space was established in Kidane-Trent [KT], it seems plausible that Wolff-type ideal results should be extended to the algebra of multipliers on weighted Dirichlet space. This is what we intend to do in this paper.

We use $\mathcal{D}_\alpha$ to denote the weighted Dirichlet space on the unit disk, $\mathbb{D}$. That is,

$$\mathcal{D}_\alpha = \{\, f:\ \mathbb{D} \to \mathbb{C} \mid\ f \text{ is analytic on } \mathbb{D} \text{ and for } f(z) = \sum_{n=0}^{\infty} a_n\, z^n,$$
$$\|f\|^2_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha \, |a_n|^2 < \infty \}.$$

We will use other equivalent norms for smooth functions in $\mathcal{D}_\alpha$ as follows,

$$\|f\|^2_{\mathcal{D}_\alpha} = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_D |f'(z)|^2 \left(1-|z|^2\right)^{1-\alpha} dA(z) \qquad \text{and}$$

$$\|f\|^2_{\mathcal{D}_\alpha} = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{1+\alpha}}\, d\sigma d\sigma.$$

For ease of notation, we will denote $\left(1-|z|^2\right)^{1-\alpha} dA(z)$ by $dA_\alpha(z)$. Also, we will consider $\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$ as an l^2-valued weighted Dirichlet space. The norms in this case are exactly as above but we will replace the absolute value by l^2-norms. Moreover, we use $\mathcal{HD}_\alpha$ to denote the harmonic weighted Dirichlet space (restricted to the boundary of $\mathbb{D}$). The functions in $\mathcal{D}_\alpha$ have only vanishing negative Fourier coefficients whereas the functions in $\mathcal{HD}_\alpha$ may have negative fourier coefficients which do not vanish. Again, if f is smooth on $\partial\mathbb{D}$, the boundary of the unit disk $\mathbb{D}$, then

$$\|f\|^2_{\mathcal{HD}_\alpha} = \int_{-\pi}^{\pi} |f|^2 \, d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{1+\alpha}}\, d\sigma d\sigma.$$

We use $\mathcal{M}(\mathcal{D}_\alpha)$ to denote the multiplier algebra of weighted Dirichlet space, defined as: $\mathcal{M}(\mathcal{D}_\alpha) = \{\phi \in \mathcal{D}_\alpha :\ \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha\}$, and we will denote the multiplier algebra of harmonic weighted Dirichlet space by $\mathcal{M}(\mathcal{HD}_\alpha)$, defined similarly (but only on $\partial\mathbb{D}$).

Given $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$, we consider $F(z) = (f_1(z),\, f_2(z), \dots)$ for $z \in \mathbb{D}$. We define the row operator $M_F^R : \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ by

$$M_F^R\left(\{h_j\}_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} f_j h_j \text{ for } \{h_j\}_{j=1}^{\infty} \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha.$$

Similarly, we define the column operator $M_F^C : \mathcal{D}_\alpha \to \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$ by

$$M_F^C\,(h) = \{f_j h\}_{j=1}^{\infty} \text{ for } h \in \mathcal{D}_\alpha.$$

We notice that $\mathcal{D}_\alpha$ is a reproducing kernel (r.k.) Hilbert space with r.k.

$$K_w(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\,(z\overline{w})^n \text{ for } z,\, w \in \mathbb{D}$$

and it is well known (see [S]) that

$$\frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} c_n (z\overline{w})^n,\ c_n > 0, \text{ for all } n.$$

Hence, weighted Dirichlet space has a reproducing kernel with "one positive square" or a "complete Nevanlinna-Pick" kernel. This property will be used to complete the first part of our proof.

We know that $\mathcal{M}(\mathcal{D}_\alpha) \subseteq H^\infty(\mathbb{D})$, but $\mathcal{M}(\mathcal{D}_\alpha) \neq H^\infty(\mathbb{D})$ (e.g., $\sum\limits_{n=1}^{\infty} \frac{z^{n^{4m+1}}}{n^{2m\alpha}}$, $m = \left[\frac{1}{\alpha}\right] + 1$, $z \in D$, is in $H^\infty(D)$ but is not in $\mathcal{D}_\alpha$ and so neither in $\mathcal{M}(\mathcal{D}_\alpha)$). Hence, $\mathcal{M}(\mathcal{D}_\alpha) \subsetneq H^\infty(\mathbb{D}) \cap \mathcal{D}_\alpha$.

Also, it is worthwhile to note that the pointwise hypothesis that $F(z)\,F(z)^\star \leq 1$ for $z \in \mathbb{D}$ implies that the analytic Toeplitz operators T_F^R and T_F^C defined on $\overset{\infty}{\underset{1}{\oplus}} H^2(\mathbb{D})$ and $H^2(\mathbb{D})$, in analogy to that of M_F^R and M_F^C, are bounded and

$$\left\|T_F^R\right\| = \left\|T_F^C\right\| = \sup_{z\in\mathbb{D}} \left(\sum_{j=1}^{\infty} |f_j(z)|^2\right)^{\frac{1}{2}} \leq 1.$$

But, since $M(\mathcal{D}_\alpha) \subsetneq H^\infty(\mathbb{D})$, the pointwise upperbound hypothesis will not be sufficient to conclude that M_F^R and M_F^C are bounded on weighted Dirichlet space. However, $\left\|M_F^R\right\| \leq \sqrt{10}\left\|M_F^C\right\|$. Thus, we will replace the natural normalization that $F(z)\,F(z)^\star \leq 1$ for all $z \in \mathbb{D}$ by the stronger condition that $\left\|M_F^C\right\| \leq 1$.

Then we have the following theorem:

Theorem 1. *Let H,$\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. Assume that*

$$\text{(a) } \|M_F^C\| \leq 1$$

$$\textit{and} \quad \text{(b) } |H(z)| \leq \sqrt{\sum_{j=1}^{\infty} |f_j(z)|^2} \quad \textit{for all } z \in \mathbb{D}.$$

Then there exists $K(\alpha) < \infty$ and there exists $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$ with

$$\|M_G^C\| \leq K(\alpha)$$
$$\textit{and} \quad F\,G^T = H^3.$$

Of course, it should be noted that for only a finite number of multipliers, $\{f_j\}$, condition (a) of Theorem 1 can always be assumed, so we have the exact analogue of Wolff's theorem in the finite case.

First, let's outline the method of our proof. Assume that $F \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha)$ and $H \in \mathcal{M}(\mathcal{D}_\alpha)$ satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant $K(\alpha) < \infty$, so that

$$M_{H^3}\, M_{H^3}^{\star} \leq K(\alpha)^2 M_F^R\, M_F^{\star R}. \tag{2}$$

Given (2), a commutant lifting theorem argument as it appears in, for example, Trent [Tr2] completes the proof by providing a $G \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha)$, so that $\|M_G^C\| \leq K(\alpha)$ and $F\,G^T = H^3$.

But (2) is equivalent to the following: there exists a constant $K(\alpha) < \infty$ so that, for any $h \in \mathcal{D}_\alpha$, there exists $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$ such that

$$\begin{aligned} &\text{(i) } M_F^R(\underline{u}_h) = H^3 h \quad \text{and} \\ &\text{(ii) } \|\underline{u}_h\|_{\mathcal{D}_\alpha} \leq K(\alpha)\, \|h\|_{\mathcal{D}_\alpha}. \end{aligned} \tag{3}$$

Hence, our goal is to show that (3) follows from (a) and (b). For this we need a series of lemmas.

Lemma 1. *Let $\{c_j\}_{j=1}^{\infty} \in l^2$ and $C = (c_1, c_2, ...) \in B(l^2, \mathbb{C})$. Then there exists Q such that the entries of Q are either 0 or $\pm c_j$ for some j and $CC^\star I - C^\star C = QQ^\star$. Also, range of Q = kernel of C.*

We will apply this lemma in our case with $C = F(z)$ for each $z \in \mathbb{D}$, when $F(z) \neq 0$. A proof of this lemma can be found in Trent [Tr2].

Given condition (b) of Theorem 1 for all $z \in \mathbb{D}$, $F \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha)$ and $H \in \mathcal{M}(\mathcal{D}_\alpha)$ with H being not identically zero, we lose no generality assuming that $H(0) \neq 0$. If $H(0) = 0$, but $H(a) \neq 0$, let $\beta(z) = \frac{a-z}{1-\bar{a}\,z}$

for $z \in \mathbb{D}$. Then since (b) holds for all $z \in \mathbb{D}$, it holds for $\beta(z)$. So we may replace H and F by $Ho\beta$ and $Fo\beta$, respectively. If we prove our theorem for $Ho\beta$ and $Fo\beta$, then there exists $G \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha)$ so that $(Fo\beta)\,G = Ho\beta$ and hence $F(Go\beta^{-1}) = H$ and $Go\beta^{-1} \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha)$, and we are done. Thus, we may assume that $H(0) \neq 0$ in (b), so $\|F(0)\|_2 \neq 0$. This normalization will let us apply some relevant lemmas from [Tr1].

It suffices to establish (i) and (ii) for any dense set of functions in $\mathcal{D}_\alpha$, so we will use polynomials. First, we will assume F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$. In this case, we write the most general solution of the pointwise problem on $\overline{\mathbb{D}}$ and find an analytic solution with uniform bounds. Then we remove the smoothness hypotheses on F and H.

For a polynomial, h, we take

$$\underline{u}_h(z) = F(z)^\star \left(F(z)F(z)^\star\right)^{-1} H^3\, h - Q(z)\underline{k}(z), \text{where } \underline{k}(z) \in l^2 \text{ for } z \in \mathbb{D}.$$

We have to find $\underline{k}(z)$ so that $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$. Thus we want $\bar{\partial}_z\, \underline{u}_h = 0$ in $\mathbb{D}$.

Therefore, we will try

$$\underline{u}_h = \frac{F^\star H^3 h}{F F^\star} - Q\left(\widehat{\frac{Q^\star F'^\star H^3 h}{(F F^\star)^2}}\right),$$

where $\widehat{k}$ is the Cauchy transform of k on $\mathbb{D}$. Note that for k smooth on $\bar{\mathbb{D}}$ and $z \in \mathbb{D}$,

$$\widehat{\underline{k}}(z) = -\frac{1}{\pi}\int_D \frac{\underline{k}(w)}{w - z}\, dA(w) \text{ and } \overline{\partial}\,\widehat{\underline{k}}(z) = k(z) \text{ for } z \in \mathbb{D}.$$

See [A] for background on the Cauchy transform.

Then it's clear that $M_F^R\left(\underline{u}_h\right) = H^3 h$ and $\underline{u}_h$ is analytic. Hence, we will be done in the smooth case if we are able to find $K(\alpha) < \infty$, only depending on α and thus independent of the polynomial, h, such that

$$\|\underline{u}_h\|_{\mathcal{D}_\alpha} \leq K(\alpha)\, \|h\|_{\mathcal{D}_\alpha} \tag{4}$$

Lemma 2. *Let $\underline{w}$ be a harmonic function on $\overline{\mathbb{D}}$, then*

$$\int_D \|Q'\underline{w}\|_{l^2}^2\, dA_\alpha \leq 8\, \|\underline{w}\|_{\mathcal{HD}_\alpha}^2.$$

Proof of this lemma can be found in [BT].

Lemma 3. *Let the operator T be defined on $L^2(\mathbb{D}, dA_\alpha)$ by*

$$(Tf)(z) = \int_D \tfrac{f(u)}{(u-z)(1-u\bar{z})} dA_\alpha,$$

for $z \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. Then

$$||Tf||^2_{A_\alpha} \leq 4\pi^2 C^2_\alpha ||f||^2_{A_\alpha},$$

where $C_\alpha = \frac{8}{\alpha^2}$.

Proof. To show that the singular integral operator, T, is bounded on $L^2(\mathbb{D}, dA_\alpha)$, we apply Zygmund's method of rotations [Z] and apply Schur's lemma an infinite number of times.

Let $f(z) = \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} a_{jk} z^j \bar{z}^k$, where $a_{ij} = 0$ except for a finite number of terms. For $z = r\, e^{i\theta}$, we relabel to get

$$f(r\, e^{i\theta}) = \sum_{l=-\infty}^{\infty} f_l(r)\, e^{il\theta}, \text{ where } f_l(r) = \sum_{k=0}^{\infty} a_{l+k\,k}\, r^{l+2k}.$$

Then

$$\|f\|^2_{A_\alpha} = \sum_{l=-\infty}^{\infty} \|f_l(r)\|_{L^2_\alpha[0,1]},$$

where the measure on $L^2_\alpha[0,1]$ is "$(1-r^2)^{1-\alpha}\, r dr$".

Now computing as in [BT], we deduce that

$$(Tf)\left(se^{it}\right) = 2\pi \sum_{l=-\infty}^{\infty} e^{i(l-1)t}\, (T_l f_l)(s),$$

$$\text{for} \quad (T_l f_l)(s) = \begin{cases} -\left(\sum\limits_{n=0}^{-l} s^{2n}\right) \int_0^1 \chi_{(0,s)}(r) \left(\frac{r}{s}\right)^{1-l} f_l(r)\, dr & \\ \quad + \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r)\ (rs)^{1-l} f_l(r)\, dr & \text{for } l \leq 0 \\ \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left(\frac{s}{r}\right)^{l-1} f_0(r)\, r dr & \text{for } l > 0. \end{cases}$$

By our construction,

$$\|Tf\|^2_{A_\alpha} = 4\,\pi^2 \sum_{l=-\infty}^{\infty} ||T_l f_l||^2_{L^2_\alpha[0,1]},$$

where the measure on $L^2[0,1]$ is "$(1-r^2)^{1-\alpha}\, r dr$". Thus, to prove our lemma it suffices to prove that

$$\sup_l \|T_l\|_{B(L^2_\alpha[0,1])} \leq C_\alpha < \infty.$$

To illustrate the technique, we show a detailed estimate for $\|T_0\|_{B(L^2_\alpha[0,1])}$. The other cases follow similarly.

Now

$$\begin{aligned}
&\int_0^1 \left|T_0 f_0(se^{it})\right|^2 (1-s^2)^{1-\alpha} s ds \\
&= 2\int_0^1\int_0^1 f_0(u) f_0(v) \left(\int_{max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha} ds}{s}\right) udu\, vdv \\
&\quad + 2\int_0^1\int_0^1 f_0(x)\, f_0(y) \left[\int_0^{min\{x,y\}} \frac{s^2\left(1-s^2\right)^{1-\alpha}}{(1-s^2)^2} sds\right] xdx\, ydy.
\end{aligned}$$

<u>*Claim(I)*</u>:

$$\begin{aligned}
&\int_0^1\int_0^1 f_0(u) f_0(v) \left(\int_{max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha} ds}{s}\right) udu\, vdv \\
&\qquad \leq \frac{25}{16}\int_0^1 |f_0(u)|^2 \left(1-u^2\right)^{1-\alpha} u\, du.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^1\int_0^1 f_0(u) f_0(v) \left(\int_{max\{u,v\}}^1 \frac{(1-s^2)^{1-\alpha} ds}{s}\right) udu\, vdv \\
&\leq \int_0^1\int_0^1 f_0(u)\, f_0(v) \left[\frac{(1-max(u^2,v^2))^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} ln\left(\frac{1}{max\{u,v\}}\right)\right] \\
&\qquad\qquad \left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha} udu\, vdv.
\end{aligned}$$

We apply Schur's Test with $p(u)=1$.

$$\begin{aligned}
&\int_0^v \left[\frac{\left(1-v^2\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} ln\left(\frac{1}{v}\right)\right]\left(1-u^2\right)^{1-\alpha} udu \\
&\qquad = \frac{1}{2}\, ln\left(\frac{1}{v^2}\right)\frac{v^2}{2} \leq \frac{1}{4}.
\end{aligned}$$

Similarly, we get $\int_v^1 \left[\frac{\left(1-u^2\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} ln\left(\frac{1}{u}\right)\right]\left(1-u^2\right)^{1-\alpha} udu \leq 1$.

Therefore,

$$\int_0^1 \left[\frac{\left(1-max(u^2,v^2)\right)^{1-\alpha}}{\left(1-u^2\right)^{1-\alpha}\left(1-v^2\right)^{1-\alpha}} ln\left(\frac{1}{max\left\{u,v\right\}}\right)\right] p(u)\left(1-u^2\right)^{1-\alpha} udu$$
$$\leq \frac{5}{4}\, p(v).$$

<u>$Claim(II)$</u>:

$$\int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\int_0^{min\{x,y\}} \frac{s^2\left(1-s^2\right)^{1-\alpha}}{(1-s^2)^2} sds\right] xdx\, ydy$$
$$\leq \frac{4}{\alpha^2}\int_0^1 |f_0(x)|^2\left(1-x^2\right)^{1-\alpha} xdx.$$

We have

$$\int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\int_0^{min\{x,y\}} \frac{s^2\left(1-s^2\right)^{1-\alpha}}{(1-s^2)^2} sds\right] xdx\, ydy$$
$$= \int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\frac{1}{2}\int_0^{min\left\{x^2,y^2\right\}} \frac{s}{(1-s)^{1+\alpha}} ds\right] xdx\, ydy$$
$$\leq \int_0^1\int_0^1 f_0(x)\, f_0(y)\left[\frac{1}{2\alpha}\frac{min\left\{x^2,y^2\right\}}{\left(1-min\left\{x^2,y^2\right\}\right)^{\alpha}}\right] xdx\, ydy.$$

For this term, we take $p(x) = \frac{1}{(1-x^2)^{\beta}}$, where $\beta = 1-\frac{\alpha}{2}$. Then, calculating, we get that

$$\int_0^y \frac{1}{2\alpha}\frac{x^2}{\left(1-x^2\right)^{\alpha+\beta}}\frac{1}{\left(1-y^2\right)^{1-\alpha}} xdx \leq \frac{1}{4\alpha\left(\beta+\alpha-1\right)}\frac{1}{\left(1-y^2\right)^{\beta}}.$$

Similarly,

$$\int_y^1 \frac{1}{2\alpha}\frac{y^2}{\left(1-y^2\right)^{\alpha}}\frac{1}{\left(1-y^2\right)^{1-\alpha}}\frac{1}{\left(1-x^2\right)^{\beta}} xdx \leq \frac{1}{4\alpha\left(\beta-1\right)}\frac{1}{\left(1-y^2\right)^{\beta}}.$$

Therefore,

$$\int_0^1 \left[\frac{1}{2\alpha}\frac{min\,\{x^2,y^2\}}{(1-min\,\{x^2,y^2\})^{\alpha}\,(1-x^2)^{1-\alpha}\,(1-y^2)^{1-\alpha}}\right] p(x)\left(1-x^2\right)^{1-\alpha} xdx$$
$$=\left(\frac{1}{4\alpha\,(\beta+\alpha-1)}+\frac{1}{4\alpha\,(1-\beta)}\right)p(y)$$
$$=\frac{1}{(4\beta+\alpha-1)\,(1-\beta)}\,p(y)=\frac{1}{\alpha^2}\,p(y).$$

Hence,

$$\int_0^1 |T_0 f_0(s)|^2\,(1-s^2)^{1-\alpha} s ds \le C_{\alpha_0}^2 \int_0^1 |f_0(s)|^2\left(1-s^2\right)^{1-\alpha} s ds,$$

where $C_{\alpha_0}=\left[\frac{5}{2}+\frac{2}{\alpha^2}\right]\le\frac{5}{\alpha^2}$.

Applying Schur's test for $l>1$ with $p(x)=\frac{1}{(1-x^2)^{\beta}}$, $\beta=1-\frac{\alpha}{2}$, we get the estimate $C_l\le\frac{5}{\alpha^2}$, independent of l. Similarly, for $l<0$ with $p(x)=1$ and $p(x)=\frac{1}{(1-x^2)^{\beta}}$, for each of the two terms, respectively, we get the estimate $C_l\le 6+\frac{2}{\alpha^2}$, independent of l . Thus we conclude that

$$\sup_l \|T_l\|_{B(L^2_\alpha[0,1])}\le\frac{8}{\alpha^2}.$$

This finishes the proof of the Lemma. □

A classical treatment of the Beurling transform can be found in Zygmund [Z]. For our purposes, we define the Beurling transform formally by

$$\mathcal{B}(\phi)=\partial_z(\widehat{\phi}),$$

where ϕ is in $C^1(\overline{\mathbb{D}})$ and $\widehat{\phi}$ is the Cauchy transform of ϕ on $\mathbb{D}$.

Lemma 4. *Let $\mathcal{B}$ denote the Beurling transform. Then*

$$||\mathcal{B}\,(f)\,||_{A_\alpha}\le\frac{23}{\alpha}\,||f||_{A_\alpha},\ f\in L^2\,(\mathbb{D},\ \mathrm{dA}_\alpha)\,.$$

Proof. To show that the Beurling transform, $\mathcal{B}$, is bounded on $L^2\,(\mathbb{D},\ \mathbb{A}_\alpha)$, we again apply Zygmund's method of rotations [Z] and apply Schur's lemma.

As in Lemma 3, we take

$$f(r\,e^{i\theta})=\sum_{l=-\infty}^{\infty} f_l(r)\,e^{il\theta},\ \text{ where }\ f_l(r)=\sum_{k=0}^{\infty} a_{l+k\,k}\,r^{l+2k}.$$

Then

$$\|f\|_{A_\alpha}^2 = \sum_{l=-\infty}^{\infty} \|f_l(r)\|_{L_\alpha^2[0,1]},$$

where the measure on $L_\alpha^2\,[0,1]$ is "$\left(1-r^2\right)^{1-\alpha} rdr$".

Now

$$\begin{aligned}
\widehat{f}(w) &= -\frac{2}{2\pi}\int_D \frac{f(z)}{z-w}\,dA(z) \\
&= 2\sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_0^{|w|} \frac{f_l(r)\ e^{i(l+n)\,\theta}}{w^{n+1}}\, r^{n+1} dr d\sigma(\theta) \\
&\quad - 2\sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_{|w|}^{1} \frac{f_l(r)\, e^{i\,(l-1-n)\,\theta} w^n}{r^n} dr d\sigma(\theta). \qquad (\star)
\end{aligned}$$

If we take $l=0$ in $(\star)$, we get that

$$\widehat{f_0}(w) = \frac{2}{w}\int_0^{|w|} f_0(r)\, rdr.$$

Therefore,

$$\begin{aligned}
\partial\widehat{f_0}(w) &= \frac{-2}{w^2}\int_0^{|w|} f_0(r)\, rdr + \frac{2}{w}\, f_0\left(|w|\right)|w|\frac{\partial(|w|)}{\partial w} \\
&= \frac{-2}{w^2}\int_0^{|w|} f_0(r)\, rdr + \frac{\overline{w}}{w}\, f_0\left(|w|\right),
\end{aligned}$$

since $\overline{w} = \frac{\partial |w|^2}{\partial w} = 2|w|\frac{\partial |w|}{\partial w},\ \ \frac{\partial |w|}{\partial w} = \frac{\overline{w}}{2|w|}$.

Thus,

$$\mathcal{B}f_0(se^{it}) = \partial\widehat{f_0}(se^{it}) = e^{-2it}\left[\frac{-2}{s^2}\int_0^{s} f_0(r)\, rdr + f_0\left(s\right)\right].$$

Similarly, a computation shows that

$$\mathcal{B}(f)(se^{it}) = \sum_{l=-\infty}^{\infty} e^{i(l-2)t}\mathcal{B}_l f_l(s),$$

$$\text{for } \mathcal{B}_l f_l(s) = \begin{cases} \dfrac{-2}{s^2}\displaystyle\int_0^s f_0(r)\, rdr + f_0\,(s) & \text{for } l = 0 \\[2ex] -2\,(l-1)s^{l-2}\displaystyle\int_s^1 \frac{f_l(r)}{r^{l-1}}dr - f_l(s) & \text{for } l \geq 1 \\[2ex] -2(1-l)s^{l-2}\int_0^s f_l(r)r^{1-l}dr + f_l(s) & \text{for } l < 0. \end{cases}$$

Thus,

$$||\mathcal{B}f||^2_{A_\alpha} = \sum_{l=-\infty}^{\infty} ||\mathcal{B}_l f_l||^2_{L^2_\alpha[0,1]},$$

where the measure on $L^2_\alpha\,[0,1]$ is "$(1-r^2)^{1-\alpha}\, rdr$".

<u>*Claim*</u>:

$$\sup_l ||\mathcal{B}_l||_{B(L^2_\alpha[0,1])} \leq \frac{23}{\alpha} < \infty.$$

Without loss of generality we may assume that $f_l(s) \geq 0$ for all l. For $l < 2$, applying Schur's test with $p(u) = 1$ or $p(u) = \frac{1}{\sqrt{u}}$, we get that $||\mathcal{B}_l||_{B(L^2[0,1])} \leq 7$. The main cases occur for $l \geq 2$. So let $l \geq 2$ be fixed. Then

$$||\mathcal{B}_l f_l||_{L^2_\alpha[0,1]} \leq 2\left(\int_0^1 |-\;(l-1)s^{l-2}\int_s^1 \frac{f_l(r)}{r^{l-1}}dr|^2(1-s^2)^{1-\alpha}sds\right)^{\frac{1}{2}} + ||f_l||_{L^2_\alpha[0,1]}$$

Now,

$$\begin{aligned} &(l-1)^2\int_0^1 s^{2(l-2)}|\int_0^1 \chi_{(s,1)}(r)\,\frac{f_l(r)}{r^{l-1}}dr|^2(1-s^2)^{1-\alpha}sds \\ &= \int_0^1\int_0^1 f_l(u)f_l(v)\left[(l-1)^2\,\frac{1}{u^l}\,\frac{1}{v^l}\,\frac{\int_0^{min\{u,v\}} s^{2(l-2)}(1-s^2)^{1-\alpha}sds}{(1-u^2)^{1-\alpha}(1-v^2)^{1-\alpha}}\right] \\ &\qquad (1-u^2)^{1-\alpha}(1-v^2)^{1-\alpha}udu\,vdv. \end{aligned}$$

Applying Schur's test with $p(u) = \frac{1}{(1-u^2)^{1-\alpha}}$, then it's sufficient to show that

$$\int_0^1\left[(l-1)^2\,\frac{1}{u^l}\,\frac{\int_0^{min\{u,v\}} s^{2l-3}(1-s^2)^{1-\alpha}ds}{(1-u^2)^{1-\alpha}}\right]udu \leq C_l\, v^l.$$

Since $(1+s)^{1-\alpha} \leq 2$ and $\frac{1}{2} \leq \frac{1}{(1+u)^{1-\alpha}} \leq 1$, we will be done if we are able to show

$$\int_0^1 \left[(l-1)^2 \, \frac{1}{u^l} \, \frac{\int_0^{min\{u,v\}} s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \leq C_l \, v^l.$$

So we are trying to prove that

$$\int_0^v \left[(l-1)^2 \, \frac{1}{u^l} \, \frac{\int_0^u s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \leq C_l v^l \quad \text{and}$$

$$\int_v^1 \left[(l-1)^2 \, \frac{1}{u^l} \, \frac{\int_0^v s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \leq C_l v^l.$$

Now

$$\begin{aligned} &\int_0^v \left[(l-1)^2 \, \frac{1}{u^l} \, \frac{\int_0^u s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}} \right] udu \\ &= \int_0^v \left[(l-1)^2 \, s^{2l-3}(1-s)^{1-\alpha} \int_s^v \frac{du}{u^{l-1} \, (1-u)^{1-\alpha}} \right] ds. \end{aligned}$$

Let $t = (1-u)^\alpha$ and change variables. Then we get that

$$\begin{aligned} &\int_0^v \left[(l-1)^2 \, s^{2l-3}(1-s)^{1-\alpha} \int_s^v \frac{du}{u^{l-1}(1-u)^{1-\alpha}} \right] ds \\ &= \int_0^v \left[\frac{1}{\alpha} \, (l-1)^2 \, s^{2l-3}(1-s)^{1-\alpha} \int_{(1-v)^\alpha}^{(1-s)^\alpha} \frac{dt}{\left(1-t^{\frac{1}{\alpha}}\right)^{(l-2)+1}} \right] ds \\ &= \int_0^v \left[\frac{1}{\alpha} \, (l-1)^2 \, s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \binom{l-2+p}{p} \int_{(1-v)^\alpha}^{(1-s)^\alpha} t^{\frac{p}{\alpha}} dt \right] ds \\ &\leq \int_0^v \left[\frac{1}{\alpha} \, (l-1)^2 \, s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(l-2+p)!}{(l-2)!\, p!} \left[\frac{((1-s)^\alpha)^{\frac{p}{\alpha}+1}}{\frac{p}{\alpha}+1} \right] \right] ds \\ &\leq \frac{2}{\alpha} \int_0^v \left[(l-1) \, s^{2l-3}(1-s)^{1-\alpha} \sum_{q=1}^{\infty} \frac{(l-3+q)!}{(l-3)!\, q!} \, \frac{(1-s)^q}{(1-s)^{1-\alpha}} \right] ds \\ &= \frac{2}{\alpha} \int_0^v \left[(l-1) \, s^{2l-3} \left(\frac{1}{(1-(1-s))^{l-3+1}} - 1 \right) \right] ds \\ &\leq \frac{2}{\alpha} \int_0^v \left[(l-1) \, s^{2l-3} \left(\frac{1}{s^{l-2}} \right) \right] ds \\ &\leq \frac{2}{\alpha} v^l. \end{aligned}$$

Now consider

$$\int_v^1 \left[(l-1)^2 \frac{1}{u^l} \frac{\int_0^v s^{2l-3}(1-s)^{1-\alpha} ds}{(1-u)^{1-\alpha}}\right] udu$$
$$= \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_v^1 \frac{du}{u^{l-1}(1-u)^{1-\alpha}}\right] ds.$$

Again, change variables with $t = (1-u)^\alpha$. So

$$\int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_v^1 \frac{du}{u^{l-1}(1-u)^{1-\alpha}}\right] ds$$
$$= \int_0^v \left[\frac{1}{\alpha}(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_0^{(1-v)^\alpha} \frac{dt}{\left(1-t^{\frac{1}{\alpha}}\right)^{l-1}}\right] ds$$
$$= \int_0^v \left[\frac{1}{\alpha}(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \binom{l-2+p}{p} \int_0^{(1-v)^\alpha} t^{\frac{p}{\alpha}} dt\right] ds$$
$$= \int_0^v \left[\frac{1}{\alpha}(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(l-3+p+1)!}{(l-2)\,(l-3)!\,p!} \left[\frac{(1-v)^{p+\alpha}}{p+1}\right]\right] ds$$
$$\leq \frac{2}{\alpha} \int_0^v \left[(l-1)\, s^{2l-3}(1-s)^{1-\alpha} \sum_{q=1}^{\infty} \frac{(l-3+q)!}{(l-3)!\,q!} \frac{(1-v)^q}{(1-v)^{1-\alpha}}\right] ds$$
$$= \frac{2}{\alpha} \int_0^v \left[(l-1)\, s^{2l-3}(1-s)^{1-\alpha} \left(\frac{1}{(1-(1-v))^{l-3+1}} - 1\right) \frac{1}{(1-v)^{1-\alpha}}\right] ds$$
$$\leq \frac{2}{\alpha} \int_0^v \left[(l-1)\, s^{2l-3}(1-s)^{1-\alpha} \left(\frac{1-v^{l-2}}{v^{l-2}}\right) \frac{(1-v)^\alpha}{(1-v)}\right] ds$$
$$\leq \frac{2}{\alpha} \int_0^v \left[(l-1)\, s^{2l-3}(1-s)^{1-\alpha}(1-s)^\alpha \left(\frac{1-v^{l-2}}{1-v}\right) \frac{1}{v^{l-2}}\right] ds$$
$$= \frac{2(l-1)}{\alpha} \int_0^v \left[\left(s^{2l-3} - s^{2l-2}\right) \left(\frac{1-v^{l-2}}{1-v}\right) \frac{1}{v^{l-2}}\right] ds$$
$$= \frac{2(l-1)}{\alpha} \left[\left(\frac{v^{2l-2}}{2l-2} - \frac{v^{2l-1}}{2l-1}\right) \left(\frac{1-v^{l-2}}{1-v}\right) \frac{1}{v^{l-2}}\right]$$
$$= \frac{2(l-1)v^l}{\alpha} \left[\left(\frac{(1-v)}{2l-2} + v\left(\frac{1}{2l-2} - \frac{1}{2l-1}\right)\right) \left(\frac{1-v^{l-2}}{1-v}\right)\right]$$
$$\leq \frac{1}{\alpha}\, v^l + \frac{2(l-1)v^{l+1}}{\alpha} \left[\left(\frac{1}{2(l-1)(2l-1)}\right) \left(\frac{1-v^{l-2}}{1-v}\right)\right]$$

$$= \frac{1}{\alpha}\ v^l + \frac{v^{l+1}}{\alpha}\left[\frac{1}{2l-1}\left(\frac{1-v^{l-2}}{1-v}\right)\right]$$
$$\leq \frac{1}{\alpha}\ v^l + \frac{v^{l+1}}{\alpha}\frac{(l-2)}{(2l-1)}$$
$$\leq \frac{2}{\alpha}\ v^l.$$

Therefore,

$$\int_0^1 \left[(l-1)^2\,\frac{1}{u^l}\,\frac{1}{v^l}\frac{\int_0^{min\left\{u^2,v^2\right\}} s^{(l-2)}(1-s)^{1-\alpha}ds}{(1-u^2)^{1-\alpha}(1-v^2)^{1-\alpha}}\right] p(u)(1-u^2)^{1-\alpha}udu$$

$$\leq \frac{4}{\alpha}\,p(v).$$

We conclude that

$$\sup_l ||\mathcal{B}_l||_{B(L^2_\alpha[0,1])} \leq 15 + \frac{8}{\alpha} \leq \frac{23}{\alpha}.$$

□

Lemma 5. *If Q is a multiplier of $\mathcal{D}_\alpha$, then*

$$\left(1-|z|^2\right)|\,Q'(z)| \leq \|M_Q\|_{B(\mathcal{D}_\alpha)} \text{ for all } z \in \mathbb{D}.$$

Proof. Define $\varphi : D \to D$ as $\varphi(z) = \frac{Q(z)}{\|M_Q\|_{B(\mathcal{D}_\alpha)}}$ for all $z \in \mathbb{D}$. Now use the Schwarz lemma and the fact that $\|\varphi\|_{\infty,\mathbb{D}} \leq \|M_\varphi\|_{\mathcal{B}(\mathcal{D}_\alpha)}$ to complete the proof. □

Lemma 6. *If $H \in \mathcal{M}\left(\mathcal{D}_\alpha\right)$, then $|H\,'|^2 dA_\alpha$ is a $\mathcal{D}_\alpha$-Carleson measure with the constant $4||M_H||^2_{B(\mathcal{D}_\alpha)}$.*

Proof. To prove the lemma, we need to show that

$$\int_{\mathbb{D}} |H\,'|^2|g|^2 dA_\alpha \leq 4||M_H||^2_{B(\mathcal{D}_\alpha)}||g||^2_{\mathcal{D}_\alpha} \text{ for all } g \in \mathcal{D}_\alpha.$$

Let $g \in \mathcal{D}_\alpha$, then

$$\begin{aligned}
\int_{\mathbb{D}} |H'|^2|g|^2 dA_\alpha &= \int_{\mathbb{D}} |\,(Hg)' - Hg'|^2 dA_\alpha \\
&\leq 2\int_{\mathbb{D}} |\,(Hg)'\,|^2 dA_\alpha + 2\int_{\mathbb{D}} |Hg'|^2 dA_\alpha \\
&\leq 2\int_{\mathbb{D}} |Hg|^2 d\sigma + 2\int_{\mathbb{D}} |\,(Hg)'\,|^2 dA_\alpha + 2\int_{\mathbb{D}} |Hg'|^2 dA_\alpha \\
&\leq 2||M_H g||^2_{\mathcal{D}_\alpha} + 2||M_H||^2_{B(\mathcal{D}_\alpha)}||g||^2_{\mathcal{D}_\alpha} \\
&\leq 4||M_H||^2_{B(\mathcal{D}_\alpha)}||g||^2_{\mathcal{D}_\alpha}.
\end{aligned}$$

□

This proves the lemma.

We are now ready to prove Theorem 1.

Proof. First, we will prove the theorem for smooth functions on $\overline{\mathbb{D}}$ and get a uniform bound. Then we will use a compactness argument to remove the smoothness hypothesis.

Assume that (a) and (b) of Theorem 1 hold for F and H and that F and H are analytic on $\mathbb{D}_{1+\epsilon}(0)$. Our main goal is to show that there exists a constant, $K(\alpha) < \infty$, independent of ϵ, so that for any polynomial, h, there exists $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha$ such that $M_F^R(\underline{u}_h) = H^3 h$ and $\|\underline{u}_h\|^2_{\mathcal{D}_\alpha} \leq K(\alpha)\,\|h\|^2_{\mathcal{D}_\alpha}$.

We take $\underline{u}_h = \frac{F^\star H^3 h}{FF^\star} - Q\Big(\widehat{\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}}\Big)$. Then $\underline{u}_h$ is analytic and $M_F^R(\underline{u}_h) = H^3 h$. We know that

$$\|\underline{u}_h\|^2_{\mathcal{D}_\alpha} = \int_{-\pi}^{\pi} \|\underline{u}_h(e^{it})\|^2\, d\sigma(t) + \int_D \|\left(\underline{u}_h(z)\right)'\|^2\, dA_\alpha(z).$$

Condition (b) implies that

$$\int_{-\pi}^{\pi} \|\frac{F^\star H^3 h}{FF^\star} - Q\left(\frac{\widehat{Q^\star F'^\star H^3} h}{(FF^\star)^2}\right)\|^2\, d\sigma(t) \leq 15\|h\|^2_\sigma \text{ (see [Tr1]).}$$

Hence, we only need to show that

$$\int_D \|\left(\frac{F^\star H^3 h}{FF^\star} - Q\left(\frac{\widehat{Q^\star F'^\star H^3} h}{(FF^\star)^2}\right)\right)'\|^2\, dA_\alpha(z) \leq K(\alpha)^2\|h\|^2_{\mathcal{D}_\alpha}$$

for some $K(\alpha) < \infty$.

Now

$$\begin{aligned}
&\int_D \left\| \left(\frac{F^\star H^3 h}{FF^\star} - Q\left(\frac{Q^\star \widehat{F'^\star H^3} h}{(FF^\star)^2} \right) \right)' \right\|^2 dA_\alpha(z) \\
&\quad \leq 5 \underbrace{\int_D \left\| \frac{F^\star 3H^2 H' h}{FF^\star} \right\|^2 dA_\alpha(z)}_{(a')} + 5 \underbrace{\int_D \left\| \frac{F^\star H^3 h'}{FF^\star} \right\|^2 dA_\alpha(z)}_{(b')} \\
&\qquad + 5 \underbrace{\int_D \left\| \frac{F^\star H^3 h' F' F^\star}{(FF^\star)^2} \right\|^2 dA_\alpha(z)}_{(c')} + 5 \underbrace{\int_D \left\| Q' \left(\frac{Q^\star \widehat{F'^\star H^3} h}{(FF^\star)^2} \right) \right\|^2 dA_\alpha(z)}_{(d')} \\
&\qquad + 5 \underbrace{\int_D \left\| Q \left(\frac{Q^\star \widehat{F'^\star H^3} h}{(FF^\star)^2} \right)' \right\|^2 dA_\alpha(z)}_{(e')}.
\end{aligned}$$

Then

$$\begin{aligned}
(a') = \int_D \left\| \frac{F^\star 3H^2 H' h}{FF^\star} \right\|^2 dA_\alpha(z) &= 9 \int_D \left\| \frac{F^\star}{\sqrt{FF^\star}} \frac{H}{\sqrt{FF^\star}} H\, H' h \right\|^2 dA_\alpha(z) \\
&\leq 9 \int_D \| H' h \|^2 dA_\alpha(z) \\
&\leq 36 \, \|M_H\|^2_{B(\mathcal{D}_\alpha)} \, \|h\|^2_{\mathcal{D}_\alpha} \quad \text{by Lemma 6.}
\end{aligned}$$

$$(b') = \int_D \left\| \frac{F^\star H^3 h'}{FF^\star} \right\|^2 dA_\alpha(z) \leq \int_D \|h'\|^2 dA_\alpha(z) \leq \|h\|^2_{\mathcal{D}_\alpha}.$$

$$\begin{aligned}
(c') = \int_D \left\| \frac{F^\star H^3 h F' F^\star}{(FF^\star)^2} \right\|^2 dA_\alpha(z) &= \int_D \left\| \frac{F^\star F' F^\star}{\sqrt{FF^\star}} \frac{H^2}{FF^\star} \frac{H}{\sqrt{FF^\star}} \, h \right\|^2 dA_\alpha(z) \\
&\leq \int_D \left\| \frac{F^\star F' F^\star}{\sqrt{FF^\star}} \, h \right\|^2 dA_\alpha(z) \\
&\leq \int_D \| F'^\star h \|^2 dA_\alpha(z) \leq 4 \, \|h\|^2_{\mathcal{D}_\alpha}.
\end{aligned}$$

We use condition (a) and Lemma 3 to estimate (e').

$$
\begin{aligned}
(e') &= \int_D \|Q\left(\widehat{\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}}\right)'\|^2\, dA_\alpha(z) \\
&\leq \int_D \|\left(\widehat{\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}}\right)'\|^2\, dA_\alpha(z) \qquad (\text{ since } \|Q(z)\|_{B(l^2)} \leq 1) \\
&\leq \left(\frac{23}{\alpha}\right)^2 \int_D \|\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}\|^2\, dA_\alpha(z) \qquad (\text{by Lemma 4}) \\
&\leq 4\left(\frac{23}{\alpha}\right)^2 \|h\|^2_{\mathcal{D}_\alpha}.
\end{aligned}
$$

So we only need estimate (d'). For this, we have

$$
\int_D \|Q'\left(\widehat{\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}}\right)\|^2\, dA_\alpha(z) = \int_D \|Q'\widehat{w}\|^2\, dA_\alpha(z),
$$

where $\widehat{w} = \left(\widehat{\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}}\right)$ is a smooth function on $\overline{\mathbb{D}}$.
Therefore,

$$
\int_D \|Q'\,\widehat{w}\|^2\, dA_\alpha(z) \leq 2\underbrace{\int_D \|Q'\widehat{w} - Q'\widetilde{\widehat{w}}\|^2\, dA_\alpha(z)}_{(f')} + 2\int_D \|Q'\widetilde{\widehat{w}}\|^2\, dA_\alpha(z),
$$

where $\widetilde{\widehat{w}}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|}\,\widehat{w}(e^{it})\, d\sigma(t)$ is the harmonic extension of $\widehat{w}$ from $\partial\mathbb{D}$ to $\mathbb{D}$.

Lemma 2 tells us that

$$
\int_D \|Q'\widetilde{\widehat{w}}\|^2\, dA_\alpha(z) \leq 8\,\|\widetilde{\widehat{w}}\|^2_{\mathcal{HD}_\alpha}.
$$

Also, Lemmas 10 and 11 of [KT] imply that there is a $C_1 < \infty$, independent of w and α, satisfying

$$
\|\underline{\widetilde{\widehat{w}}}\|^2_{\mathcal{HD}_\alpha} \leq C_1 \|w\|^2_{A_\alpha}.
$$

But, as we showed above

$$
\|w\|^2_{A_\alpha} = \int_D \|\frac{Q^\star F'^\star H^3 h}{(FF^\star)^2}\|^2\, dA_\alpha(z) \leq \int_D \|F'^\star h\|^2\, dA_\alpha(z) \leq 4\,\|h\|^2_{\mathcal{D}_\alpha}.
$$

Thus,

$$
\int_D \|Q'\widetilde{\widehat{w}}\|^2\, dA_\alpha(z) \leq C_2\,\|h\|^2_{\mathcal{D}_\alpha},
$$

where $C_2 < \infty$ is independent of w and α.

Now we are just left with estimating (f'). We have

$$
\begin{aligned}
(f') &= \int_D \|Q^{'}\widehat{w} - Q^{'}\widetilde{\widehat{w}}\|^2 \, dA_\alpha(z) \\
&= \int_D \|Q^{'}\Big[-\frac{1}{\pi}\int_D \frac{w(u)}{u-z} dA(u) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|}\widehat{w}(e^{it}) d\sigma(t)\Big] \|^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2}\int_D \|Q^{'}\int_D w(u)\Big[\frac{1}{u-z} + \int_{-\pi}^{\pi}\frac{1-|z|^2}{|1-e^{-it}z|}\, e^{-it}\frac{1}{1-ue^{-it}}\, d\sigma(t)\Big] \\
&\qquad\qquad dA(u)\|^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2}\int_D \|Q^{'}\int_D w(u)\left[\frac{1}{u-z} + \frac{\bar{z}}{1-u\bar{z}}\right] dA(u)\|^2 \, dA_\alpha(z) \\
&= \frac{1}{\pi^2}\int_D \|Q^{'}\int_D w(u)\left[\frac{1-|z|^2}{(u-z)(1-u\bar{z})}\right] dA(u)\|^2 \, dA_\alpha(z) \\
&= \frac{1}{\pi^2}\int_D \|Q'(z)\,(1-|z|^2)\, T(w)(z)\|^2 \, dA_\alpha(z) \\
&\le \frac{\|M_Q\|^2}{\pi^2}\|T(w)\|^2_{A_\alpha} \qquad \text{by Lemma 5} \\
&\le \frac{256}{\alpha^4}\|M_Q\|^2\,\|w\|^2_{A_\alpha} \quad \text{by Lemma 3.} \\
&\le \frac{1024}{\alpha^4}\|M_Q\|^2\,\|h\|^2_{\mathcal{D}_\alpha}
\end{aligned}
$$

By Lemma 9 of [KT], we have $\|M_Q\|_{B(\oplus\mathcal{D}_\alpha)} \le \sqrt{86}$. Combining all these pieces, we see that in the smooth case

$$\|\underline{u}_h\|^2_{\mathcal{D}_\alpha} \le K(\alpha)^2\,\|h\|^2_{\mathcal{D}_\alpha}\,,$$

where $K(\alpha) = K_1\|M_H\|_{B(\mathcal{D}_\alpha)} + \frac{K_2}{\alpha^2}$, where $K_1 < \infty$ and $K_2 < \infty$ are constants independent of h,ϵ and α.

By the proof of Theorem 1 in the smooth case, we have

$$M^R_{F_r}(M^R_{F_r})^\star \le K(\alpha)^2 M_{H_r}M^\star_{H_r} \text{ for } 0 \le r < 1.$$

Using a commutant lifting argument, there exists $G_r \in \mathcal{M}(\mathcal{D}_\alpha, \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}_\alpha)$ so that $M^R_{F_r}M^C_{G_r} = M_{H^3_r}$ and $\|M^R_{G_r}\| \le K(\alpha)$. Then $M^R_{F_r} \to M^R_F$ and $M_{H_r} \to M_H$ as $r \uparrow 1$ in the $\star-$strong topology.

By compactness, we may choose a net with $G^\star_{r_\alpha} \to G^\star$ as $r_\alpha \to 1^-$. Since the multiplier algebra (as operators) is WOT closed, $G \in$

$\mathcal{M}(\mathcal{D}_\alpha, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha)$. Also, since $F^\star_{r_\alpha} \overset{s}{\to} F^\star$, we get $M^\star_{H_r} = M^{\star C}_{G_r} M^{\star R}_{F_r} \overset{WOT}{\to} M^{\star C}_G M^{\star R}_F$ and so $M^R_F M^C_G = M_{H^3}$ with entries of G in $\mathcal{M}(\mathcal{D}_\alpha)$ and $\|M^C_G\| \le K(\alpha)$.

This ends our proof. $\square$

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